Refresher: Untyped λ-calculus

Syntax: Terms

$$t ::= x \qquad \text{Variable}$$
$$\mid \lambda x . t \qquad \text{Abstraction}$$
$$\mid (t)\ t \qquad \text{Application}$$

Bound variables:

$$bv(x) = \emptyset$$
$$bv(\lambda x . t) = bv(t) \cup \{x\}$$
$$bv((t)s) = bv(t) \cup bv(s)$$

Free variables:

$$fv(x) = \{x\}$$
$$fv(\lambda x . t) = fv(t) - \{x\}$$
$$fv((t)s) = fv(t) \cup fv(s)$$

α-conversion: $\lambda x . t \equiv_\alpha \lambda y . t[x := y]$ if $y \notin fv(t)$

β-reduction: $(\lambda x . t)\ s \longrightarrow_\beta t[x := s]$

η-conversion: $\lambda x . (t)\ x \equiv_\eta t$ if $x \notin fv(t)$

Ex 1:

(a) No: any abstraction definition requires a lambda: $\lambda x.y$

(x) $x.x$

(b) Yes: correct abstraction definitions

(c) Yes: correct abstraction and application

(d) No: we require parentheses around the 'function' in any application

$\lambda x.(x)\ x)$

(e) Yes: correct abstraction and application

Ex 2:

(a) $(\lambda x.(x)\ x)\ \lambda x.x$

$\rightarrow_\beta (\lambda x.x)\ \lambda x.x$

$\rightarrow_\beta \lambda x.x$

(b) $((\lambda x.\lambda y.(y)\ x)\ f)\ \lambda x.x$

$\rightarrow_\beta (\lambda y.(y)\ f)\ \lambda x.x$

$\rightarrow_\beta (\lambda x.x)\ f$

$\rightarrow_\beta f$

(c) $((\lambda f.\ \lambda x.\ (f)\ (f)\ x)\ \lambda x.(tall)\ x)\ student$

$\equiv_\eta\ ((\lambda f.\ \lambda x.\ (f)\ (f)\ x)\ tall)\ student$

$\rightarrow_\beta (\lambda x.\ (tall)\ (tall)\ x)\ student$

$\rightarrow_\beta\ (tall)\ (tall)\ student$

(d) $(((compose)\ \lambda x.\ (not)\ x)\ \lambda y.\ (tired)\ y)\ john$

$\equiv_\eta\ (((compose)\ not)\ tired)\ john$

$\rightarrow_\beta ((\lambda g.\ \lambda x.\ (not)\ (g)\ x)\ tired)\ john$

$\rightarrow_\beta (\lambda x.\ (not)\ (tired)\ x)\ john$

$\rightarrow_\beta\ (not)\ (tired)\ john$

(e) $((and\text{-}pred)\ \lambda x.\ (smart)\ x)\ \lambda y.\ (kind)\ y$

$\rightarrow_\beta (\lambda Q.\ \lambda z.\ (\lambda x.\ (smart)\ x)\ z\ \wedge\ (Q)\ z)\ \lambda y.(kind)\ y$

$\rightarrow_\beta (\lambda Q.\ \lambda z.\ (smart)\ z\ \wedge\ (Q)\ z)\ \lambda y.\ (kind)\ y$

$\rightarrow_\beta\ \lambda z.\ (smart)\ z\ \wedge\ (\lambda y.\ (kind)\ y)\ z)$

$\rightarrow_\beta\ \lambda z.\ (smart)\ z\ \wedge\ (kind)\ z$

Ex 3:

$$((\lambda S.\lambda V.(S)(V)\ \lambda Q.(Q)\,m)\ \lambda P.(P)\,j)\ \lambda O.\lambda y.(O)\,\lambda z.((kiss)\,y)\,z$$

$$\to_\beta (\lambda V.(\lambda P.(P)\,j)(V)\ \lambda Q.(Q)\,m)\ \lambda O.\lambda y\,(O)\,\lambda z.((kiss)\,y)\,z$$

$$\to_\beta (\lambda P.(P)\,j)\ (\lambda O.\lambda y.(O)\,\lambda z.((kiss)\,y)\,z)\ \lambda Q.(Q)\,m$$

$$\to_\beta (\lambda P.(P)\,j)\ \lambda y.(\lambda Q.(Q)\,m)\ \lambda z.((kiss)\,y)\,z)$$

$$\to_\beta (\lambda y.(\lambda Q.(Q)\,m)\ \lambda z.((kiss)\,y)\,z)\ j$$

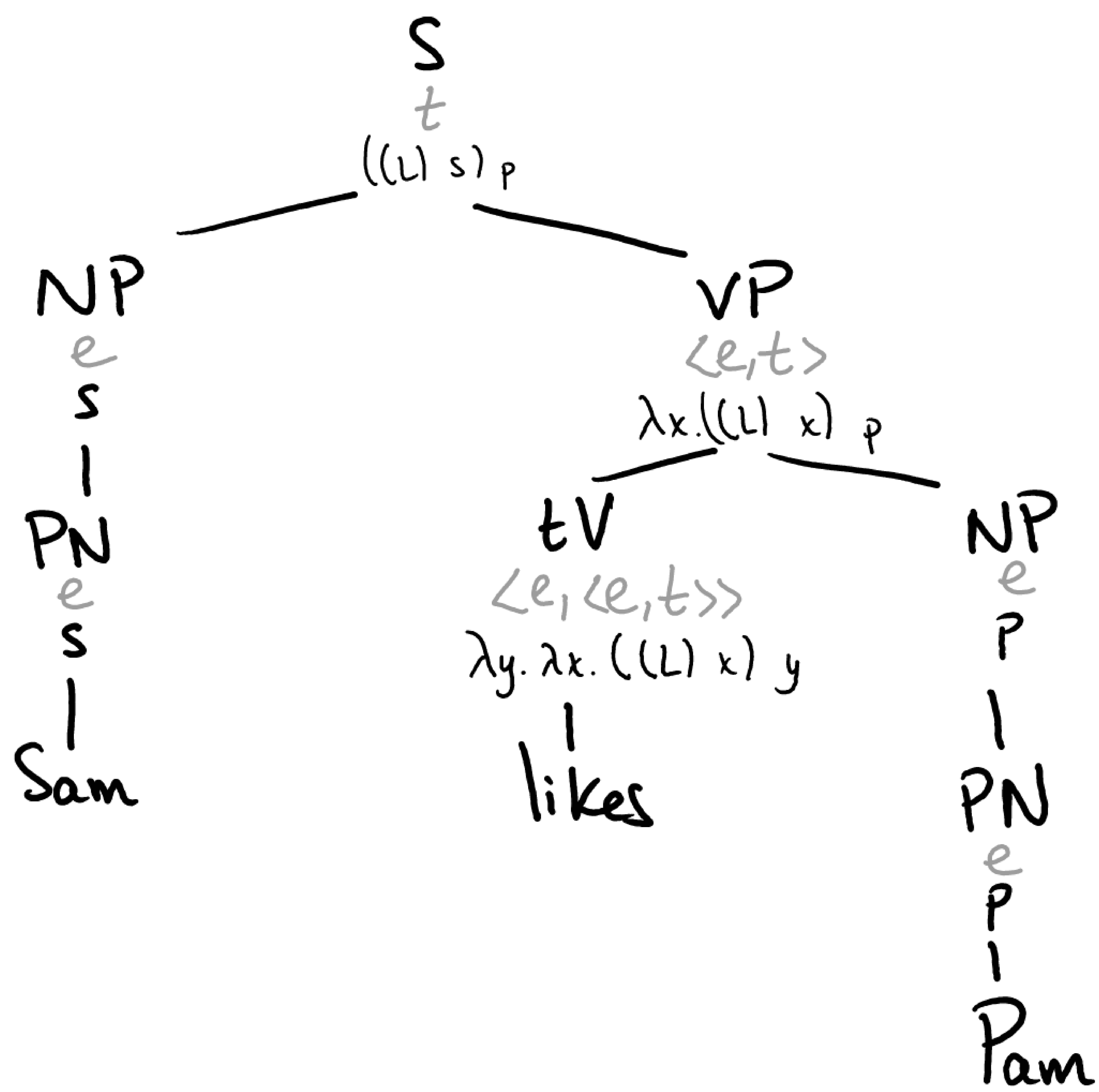
$$\to_\beta (\lambda Q.(Q)\,m)\ \lambda z.((kiss)\,j)\,z)$$

$$\to_\beta (\lambda z.((kiss)\,j)\,z)\ m$$

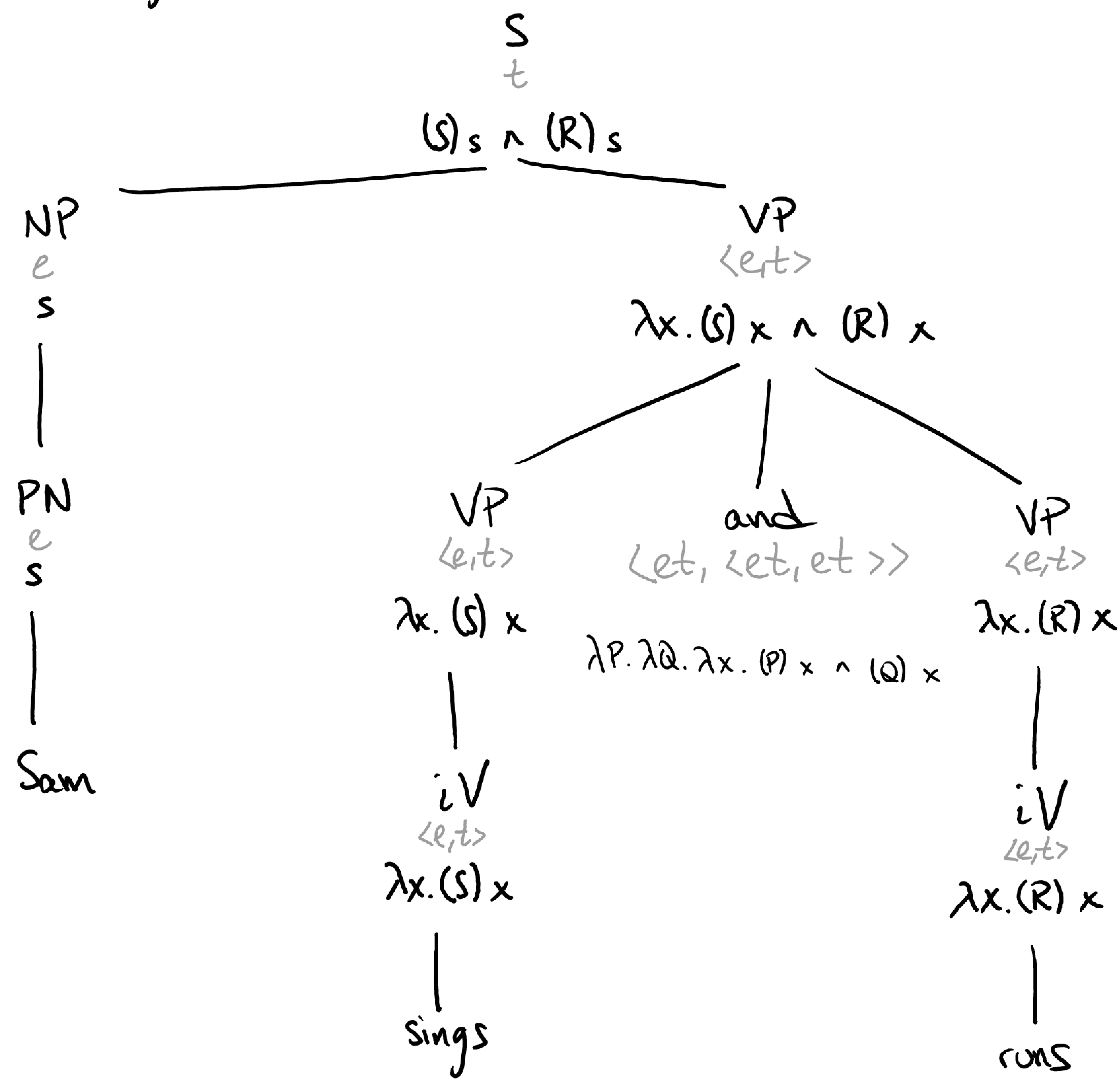
$$\to_\beta ((kiss)\,j)\,m$$

Ex 4:

(a) Sam likes Pam.



(b) Sam sings and runs.



$$[\![VP]\!] = \left(\left([\![and]\!] \right) [\![VP]\!] \right) [\![VP]\!]$$

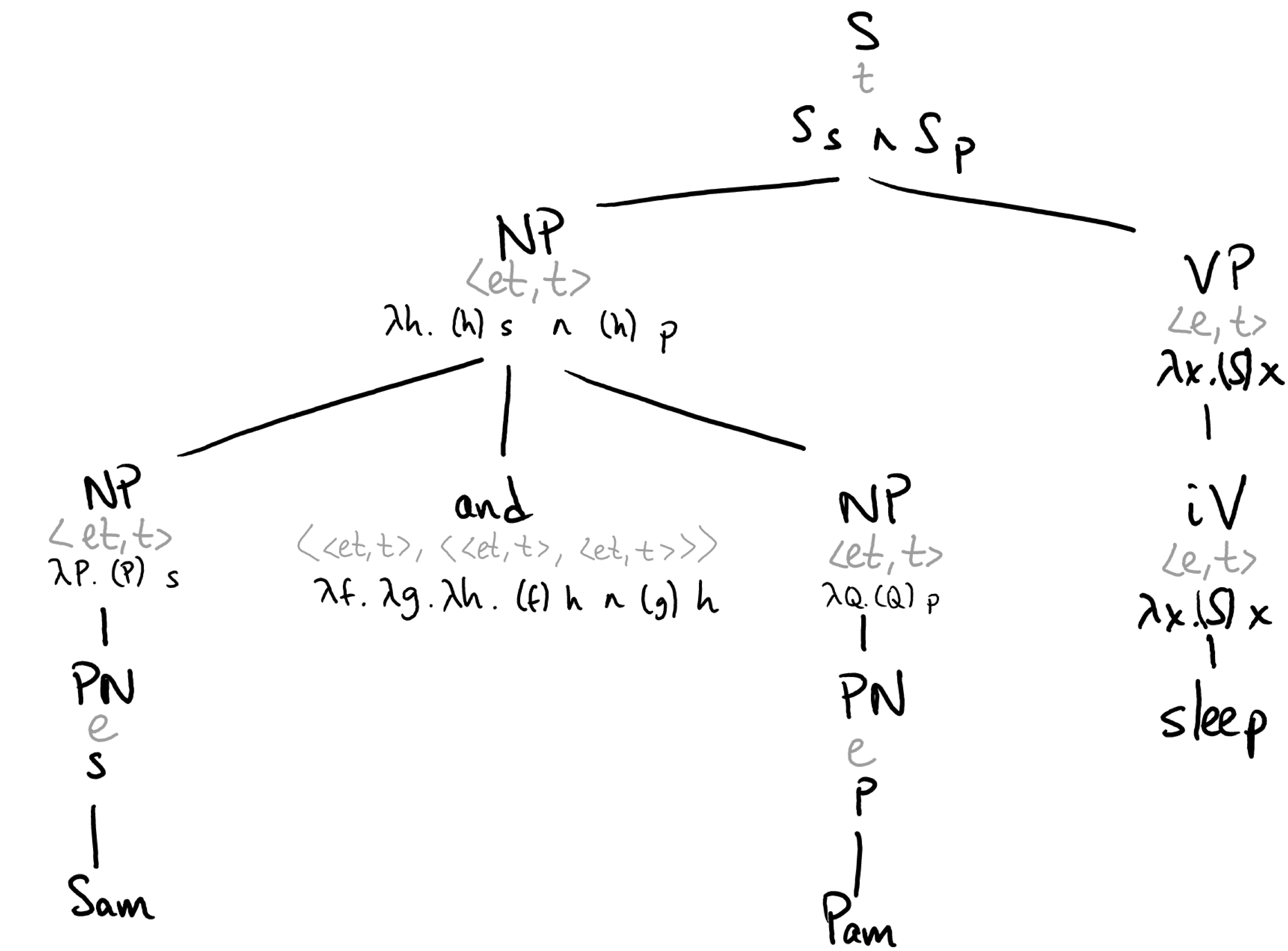
$$= \left((\lambda P.\lambda Q.\lambda x.(P)\,x \wedge (Q)\,x)\ \lambda x.(S)\,x \right)\ \lambda x.(R)\,x$$

$$= \lambda Q.\lambda x.\ (\lambda y.(S)\,y)\,x \wedge (Q)\,x)\ \lambda x.(R)\,x$$

$$= \lambda x.\ (\lambda y.(S)\,y)\,x \wedge (\lambda y.(R)\,y)\,x$$

$$= \lambda x.(S)\,x \wedge (R)\,x$$

(c) Sam and Pam sleep.



$$\llbracket NP \rrbracket = ((\llbracket and \rrbracket)\ \llbracket NP \rrbracket)\ \llbracket NP \rrbracket$$

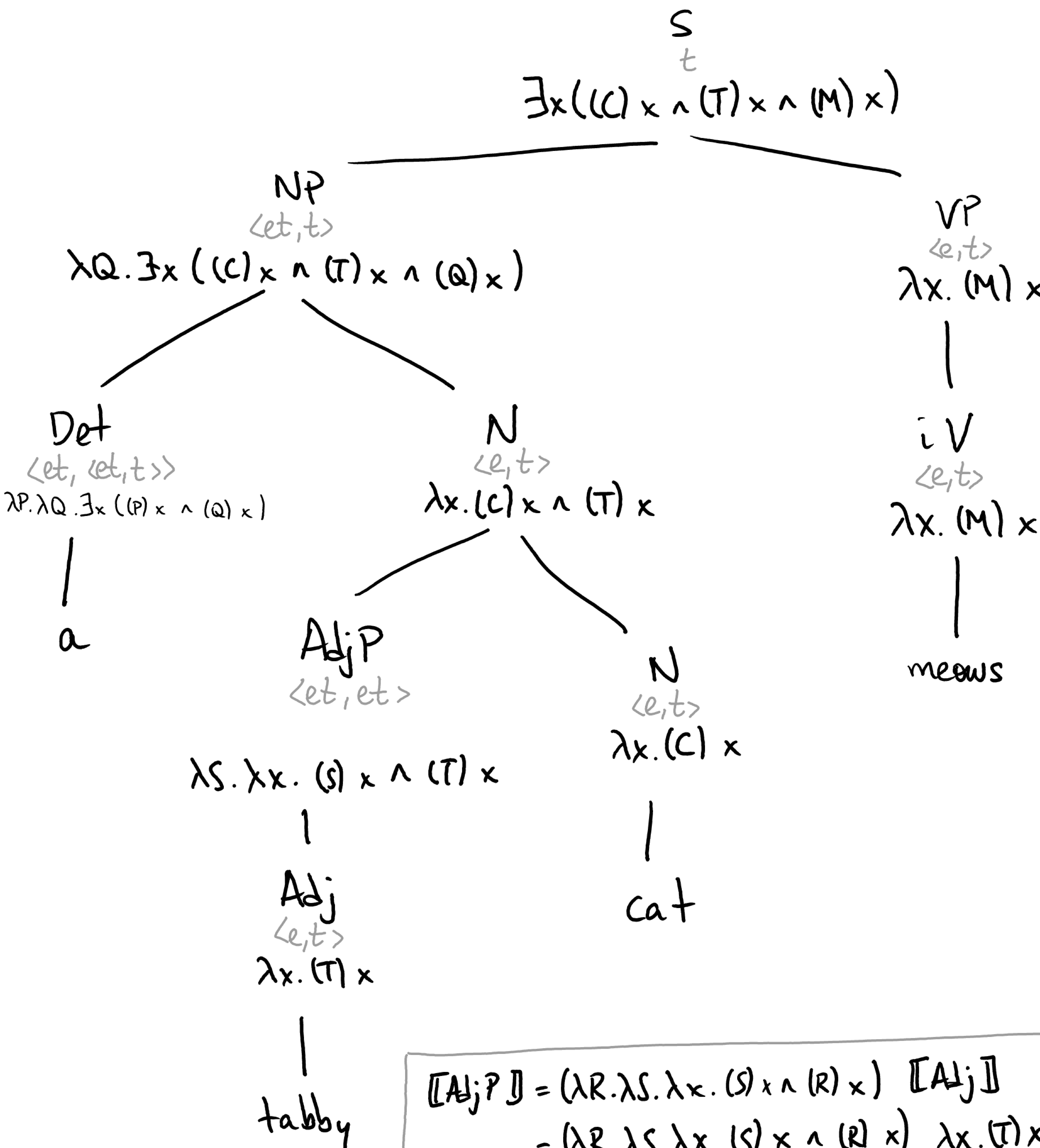
$$= (((\lambda f.\lambda g.\lambda h.\ (f)\ h \wedge (g)\ h)\ \lambda P.(P)\ s)\ \lambda Q.(Q)\ p$$

$$= (\lambda g.\lambda h.\ (\lambda P.(P)\ s)\ h \wedge (g)\ h)\ \lambda Q.(Q)\ p$$

$$= \lambda h.\ (\lambda P.(P)\ s)\ h \wedge (\lambda Q.(Q)\ p)\ h)$$

$$= \lambda h.\ (h)\ s \wedge (h)\ p$$

(d) A tabby cat meows.



$$
\begin{aligned}
[\![AdjP]\!] &= (\lambda R.\lambda S.\lambda x.(S)\,x \wedge (R)\,x)\ [\![Adj]\!] \\
&= (\lambda R.\lambda S.\lambda x.(S)\,x \wedge (R)\,x)\ \lambda x.(T)\,x \\
&= \lambda S.\lambda x.(S)\,x \wedge (\lambda y.(T)\,y)\,x) \\
&= \lambda S.\lambda x.(S)\,x \wedge (T)\,x
\end{aligned}
$$