First Order Logic Language

(i) If A is a predicate constant, of arity n, and each $t_1 ... t_n$ an individual constant or variable, then $A(t_1, ..., t_n)$ is a wff.
(ii) If φ is a wff, then so is $\neg\varphi$.
(iii) If φ and ψ are wffs, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$.
(iv) If φ is a wff and x a variable, then $\forall x\varphi$ and $\exists x\varphi$ are wffs.
(v) Nothing else is a wff.

Scope

If $\forall x\psi$ is a sub-formula of φ, then ψ is called the **scope** of this occurrence of the quantifier $\forall x$ in φ. Same definition for $\exists x$.

Bound/Free variable

(a) An occurrence of a variable x in the formula ϕ (which is not part of a quantifer) is called **free** if this occurrence of x is not in the scope of a quantifier $\forall x$ ou $\exists x$ occurring in ϕ.
(b) If $\forall x\psi$ (or $\exists x\psi$) is a sub-formula of ϕ and x is free in ψ, then this occurrence of x is called **bound** by the quantifier $\forall x$ (or $\exists x$).

A **sentence** is a formula with no free variable.

Tarskian truth definition Let $[\![\alpha]\!]^g_{\mathcal{M}}$ be the denotation of α in the model $\mathcal{M} = \langle D, I\rangle$ and with the assignment g.

$[\![t]\!]^g_{\mathcal{M}} = I(t)$ if t is an individual constant
$[\![t]\!]^g_{\mathcal{M}} = g(t)$ if t is a variable

$$[\![P(t_1, ...t_n)]\!]^g_{\mathcal{M}} = 1 \text{ iff } \langle [\![t_1]\!]^g_{\mathcal{M}}, ...[\![t_n]\!]^g_{\mathcal{M}}\rangle \in I(P).$$

If φ and ψ are wfss,

$$\begin{array}{lll} [\![\neg\varphi]\!]^g_{\mathcal{M}} = 1 & \text{iff} & [\![\varphi]\!]^g_{\mathcal{M}} = 0 \\ [\![(\varphi \wedge \psi)]\!]^g_{\mathcal{M}} = 1 & \text{iff} & [\![\varphi]\!]^g_{\mathcal{M}} = 1 \text{ and } [\![\psi]\!]^g_{\mathcal{M}} = 1 \\ [\![(\varphi \vee \psi)]\!]^g_{\mathcal{M}} = 1 & \text{iff} & [\![\varphi]\!]^g_{\mathcal{M}} = 1 \text{ or } [\![\psi]\!]^g_{\mathcal{M}} = 1 \\ [\![(\varphi \rightarrow \psi)]\!]^g_{\mathcal{M}} = 1 & \text{iff} & [\![\varphi]\!]^g_{\mathcal{M}} = 0 \text{ or } [\![\psi]\!]^g_{\mathcal{M}} = 1 \end{array}$$

$$\boxed{[\![\exists y\ \varphi]\!]^g_{\mathcal{M}} = 1 \text{ iff there is a } d \in D \text{ s.t. } [\![\varphi]\!]^{g[y/d]}_{\mathcal{M}} = 1}$$

similarly,

$$\boxed{[\![\forall y\ \varphi]\!]^g_{\mathcal{M}} = 1 \text{ iff for all } d \in D,\ [\![\varphi]\!]^{g[y/d]}_{\mathcal{M}} = 1}$$

If φ is a sentence :

$$\underline{[\![\varphi]\!]_{\mathcal{M}} = 1 \text{ iff there is an assignment } g \text{ such that } [\![\varphi]\!]^g_{\mathcal{M}} = 1}$$